

Conceptual Category for Grades 9 – 12: **Number and Quantity**

Domains	The Real Number System	Quantities	The Complex Number System	Vector and Matrix Quantities
Clusters	<ul style="list-style-type: none"> Extend the properties of exponents to rational exponents Use properties of rational and irrational numbers 	<ul style="list-style-type: none"> Reason quantitatively and use units to solve problems 	<ul style="list-style-type: none"> Perform arithmetic operations with complex numbers Represent complex numbers and their operations on the complex plane Use complex numbers in polynomial identities and equations 	<ul style="list-style-type: none"> Represent and model with vector quantities. Perform operations on vectors. Perform operations on matrices and use matrices in applications.
Mathematical Practices	<ol style="list-style-type: none"> 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 	<ol style="list-style-type: none"> 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 	<ol style="list-style-type: none"> 5. Use appropriate tools strategically. 6. Attend to precision. 	<ol style="list-style-type: none"> 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

Numbers and Number Systems. During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, “number” means “counting number”: 1, 2, 3... Soon after that, 0 is used to represent “none” and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers. With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties and their new meanings are consistent with their previous meanings. Extending the properties of whole-number exponents leads to new and productive notation. For example, properties of whole-number exponents suggest that $(5^{1/3})^3$ should be $5^{(1/3)3} = 5^1 = 5$ and that $5^{1/3}$ should be the cube root of 5. Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. They can be used to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

Quantities. In real world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly “stands out” as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.

Conceptual Category for Grades 9 – 12: **Algebra**

Domains	Seeing Structure in Expressions	Arithmetic with Polynomials and Rational Expressions	Creating Equations	Reasoning with Equations and Inequalities
Clusters	<ul style="list-style-type: none"> Interpret the structure of expressions Write expressions in equivalent forms to solve problems 	<ul style="list-style-type: none"> Perform arithmetic operations on polynomials Understand the relationship between zeros and factors of polynomials Use polynomial identities to solve problems Rewrite rational expressions 	<ul style="list-style-type: none"> Create equations that describe numbers or relationships 	<ul style="list-style-type: none"> Understand solving equations as a process of reasoning and explain the reasoning Solve equations and inequalities in one variable Solve systems of equations Represent and solve equations and inequalities graphically
Mathematical Practices	1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively.	3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics.	5. Use appropriate tools strategically. 6. Attend to precision.	7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

Expressions. An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances. Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p . Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor. Algebraic manipulations are governed by the properties of operations and exponents, and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, $p + 0.05p$ is the sum of the simpler expressions p and $0.05p$. Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure. A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

Equations and inequalities. An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form. The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system. An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions. Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers. The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ((b_1 + b_2)/2)h$, can be solved for h using the same deductive process. Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

Connections to Functions and Modeling. Expressions can define functions, and equivalent expressions define the same function. Asking when two functions have the same value for the same input leads to an equation; graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.

Conceptual Category for Grades 9 – 12: **Functions**

Domains	Interpreting Functions	Building Functions	Linear, Quadratic, and Exponential Models	Trigonometric Functions
Clusters	<ul style="list-style-type: none"> Understand the concept of a function and use function notation Interpret functions that arise in applications in terms of the context Analyze functions using different representations 	<ul style="list-style-type: none"> Build a function that models a relationship between two quantities Build new functions from existing functions 	<ul style="list-style-type: none"> Construct and compare linear, quadratic, and exponential models and solve problems Interpret expressions for functions in terms of the situation they model 	<ul style="list-style-type: none"> Extend the domain of trigonometric functions using the unit circle Model periodic phenomena with trigonometric functions Prove and apply trigonometric identities
Mathematical Practices	<ol style="list-style-type: none"> 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 	<ol style="list-style-type: none"> 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 	<ol style="list-style-type: none"> 5. Use appropriate tools strategically. 6. Attend to precision. 	<ol style="list-style-type: none"> 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

Functions describe situations where one quantity determines another. For example, the return on \$10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models. In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v ; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is T . The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context. A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties. Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships. A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling, and Coordinates. Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.

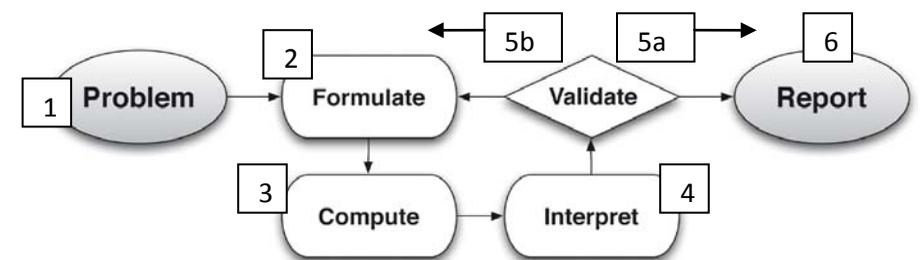
Conceptual Category for Grades 9 – 12: **Modeling**

Domains	<i>Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (☆).</i>							
Clusters								
Mathematical Practices	1. Make sense of problems and persevere in solving them.	2. Reason abstractly and quantitatively.	3. Construct viable arguments and critique the reasoning of others.	4. Model with mathematics.	5. Use appropriate tools strategically.	6. Attend to precision.	7. Look for and make use of structure.	8. Look for and express regularity in repeated reasoning.

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data. A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations. One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function. The basic modeling cycle is summarized in the diagram. It involves

- (1) identifying variables in the situation and selecting those that represent essential features,
- (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables,
- (3) analyzing and performing operations on these relationships to draw conclusions,
- (4) interpreting the results of the mathematics in terms of the original situation,
- (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if is acceptable,
- (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.



In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO2 over time. Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems. Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

Conceptual Category for Grades 9 – 12: **Geometry**

Domains	Congruence	Similarity, Right Triangles, and Trigonometry	Circles	Expressing Geometric Properties with Equations	Geometric Measurement and Dimension	Modeling with Geometry		
Clusters	<ul style="list-style-type: none"> Experiment with transformations in the plane Understand congruence in terms of rigid motions Prove geometric theorems Make geometric constructions 	<ul style="list-style-type: none"> Understand similarity in terms of similarity transformations Prove theorems involving similarity Define trigonometric ratios and solve problems involving right triangles Apply trigonometry to general triangles 	<ul style="list-style-type: none"> Understand and apply theorems about circles Find arc lengths and areas of sectors of circles 	<ul style="list-style-type: none"> Translate between the geometric description and the equation for a conic section Use coordinates to prove simple geometric theorems algebraically 	<ul style="list-style-type: none"> Explain volume formulas and use them to solve problems Visualize relationships between two dimensional and three-dimensional objects 	<ul style="list-style-type: none"> Apply geometric concepts in modeling situations 		
Mathematical Practices	1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively.		3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics.		5. Use appropriate tools strategically. 6. Attend to precision.		7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.	

- An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.
- Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)
- During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.
- The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.
- In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

- Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.
- The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to nonright triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.
- Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.
- Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations. The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.

Conceptual Category for Grades 9 – 12: **Statistics and Probability**

Domains	Interpreting Categorical and Quantitative Data	Making Inferences and Justifying Conclusions	Conditional Probability and the Rules of Probability	Using Probability to Make Decisions
Clusters	<ul style="list-style-type: none"> Summarize, represent, and interpret data on a single count or measurement variable Summarize, represent, and interpret data on two categorical and quantitative variables Interpret linear models 	<ul style="list-style-type: none"> Understand and evaluate random processes underlying statistical experiments Make inferences and justify conclusions from sample surveys, experiments and observational studies 	<ul style="list-style-type: none"> Understand independence and conditional probability and use them to interpret data Use the rules of probability to compute probabilities of compound events in a uniform probability model 	<ul style="list-style-type: none"> Calculate expected values and use them to solve problems Use probability to evaluate outcomes of decisions
Mathematical Practices	<ol style="list-style-type: none"> 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 	<ol style="list-style-type: none"> 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 	<ol style="list-style-type: none"> 5. Use appropriate tools strategically. 6. Attend to precision. 	<ol style="list-style-type: none"> 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account. Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken. Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn. Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables. Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

Connections to Functions and Modeling. Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.